

# Formation of fundamental structures in Bose-Einstein Condensates

T. F. Scott<sup>†</sup>, R. J. Ballagh<sup>†</sup>, and K. Burnett<sup>‡</sup>

<sup>†</sup> Department of Physics, University of Otago, P.O.Box 56, Dunedin, New Zealand

<sup>‡</sup> Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road,  
Oxford OX1 3PU, United Kingdom.

## Abstract.

The meanfield interaction in a Bose condensate provides a nonlinearity which can allow stable structures to exist in the meanfield wavefunction. We discuss a number of examples where condensates, modelled by the one dimensional Gross Pitaevskii equation, can produce gray solitons and we consider in detail the case of two identical condensates colliding in a harmonic trap. Solitons are shown to form from dark interference fringes when the soliton structure, constrained in a defined manner, has lower energy than the interference fringe and an analytic expression is given for this condition.

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The experimental realisation of atomic gas Bose Einstein condensates has opened the door to the study of quantum phenomena on a mesoscopic scale, and a major interest is whether qualitatively new phenomena will occur. Some of the most striking experimental results to date involve bulk dynamics of the condensate, and the JILA and MIT groups have made quantitative measurements of collective excitations [1], sound propagation [2] and interaction between distinct condensates [3, 4]. A beautiful demonstration of quantum interference has been made by Ketterle's group, where they allowed two distinct condensates (in the same internal state) to freely expand into each other to produce fringes [5]. Careful analysis of this experiment [6] has shown that the simple picture of linear interference does not adequately describe the results, and an additional mechanism, collisional self interaction, plays an important role in the formation of the fringe pattern. In this paper we describe and analyse a noteworthy new phenomenon arising in condensates as a result of the self interaction, namely the formation of fundamental structures.

It is now well accepted that the Gross-Pitaevskii (GP) equation

$$i\frac{\partial\Psi(\mathbf{r},\tau)}{\partial\tau} = -\nabla^2\Psi(\mathbf{r},\tau) + V(\mathbf{r})\Psi(\mathbf{r},\tau) + C|\Psi(\mathbf{r},\tau)|^2\Psi(\mathbf{r},\tau), \quad (1)$$

provides a good description for the low temperature behaviour of a condensate in a single internal state. In this equation, (in which dimensionless units are used as in Ruprecht *et al*[8]),  $\Psi$  can be interpreted as the condensate wavefunction [9],  $V(\mathbf{r})$  is the trap potential, and  $C$  is proportional to the number of atoms in the condensate and the scattering length. The self interaction is described by the final term of Eq.(1), introducing a nonlinearity into the description which, as is well known from other branches of physics, can profoundly alter the character of the system behaviour. A notable new feature that can appear is solitons [10], which are well known only in the one-dimensional case, but in the 2 or 3D case, analogous structures with inherent particle-like robustness, such as vortices, can also occur. Even in the 1D case, no general results are known for the GP equation. Morgan et al [11] have shown the existence of solitary wave behaviour under some well defined conditions. In this paper we present a simple scenario in which colliding trapped condensates produce a particular class of fundamental structure, gray solitons. These topological solitons appear as a density dip with an associated phase kink, and Reinhardt and Clark [7] have discussed how the gray solitons resulting from such collisions might be used to establish the initial phase of the two condensates. In this paper we focus on the mechanism of formation of the gray solitons. Using numerical solutions of Eq.(1) in one dimension, we have explored the phenomenon of gray soliton formation and identified the key mechanisms, thereby allowing us to develop an analytic description to provide quantitative understanding. The 1D case, although a considerable simplification over the more realistic 2D or 3D cases, nevertheless captures some essential characteristics including the coupling between nonlinearity and spatial variation, and we expect it will provide guidance for eventual extension into 2 or 3 spatial dimensions.

We have found that gray solitons can be produced by a variety of means, but for this paper we have chosen to study in detail the simplest scenario that enables us to identify and understand the key mechanisms involved, namely the case of two separated condensates colliding under the influence of a harmonic trap. This is similar in spirit to the interference experiment of Ketterle's group, but has some distinctive features which are important for the analysis we give. As Morgan et al [11] have shown, a spatially displaced ground state eigenstate of the time independent GP equation,

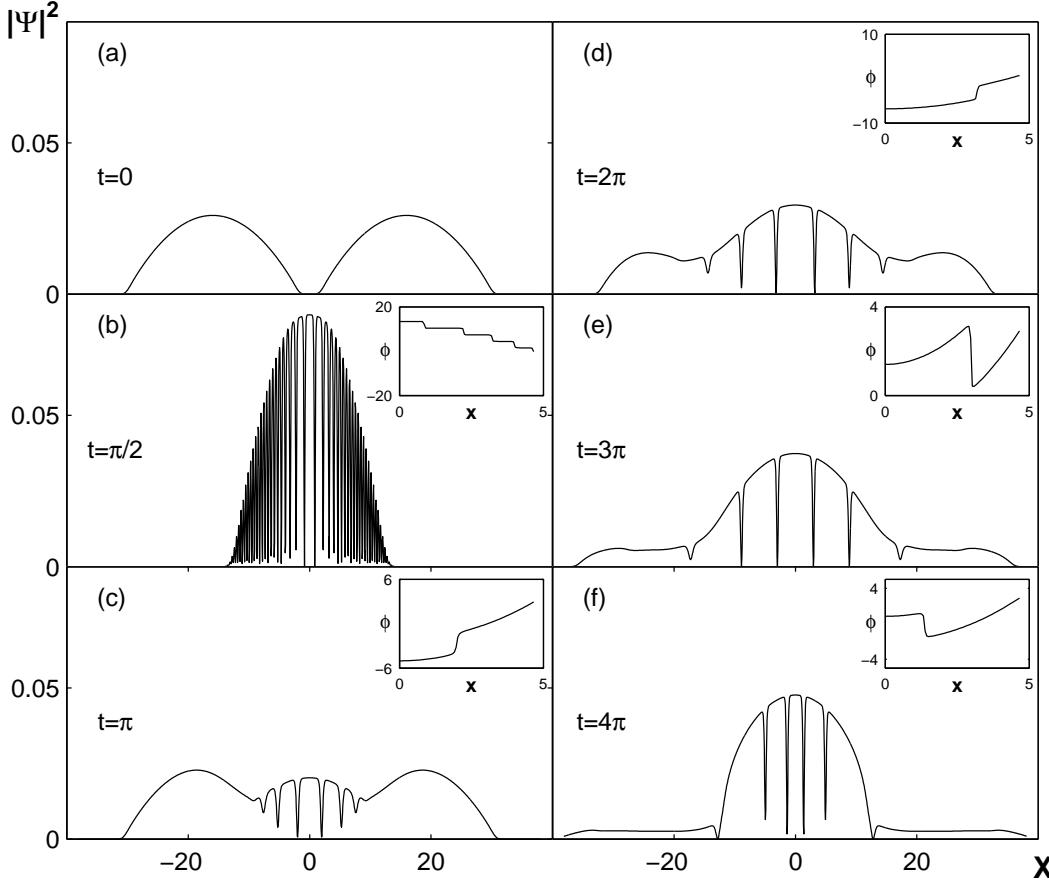
evolving alone in the trap, retains its shape while executing simple harmonic motion with period  $\tau = 2\pi$  in the present units. The spatial phase profile is then always linear in the spatial variable  $x$ , with a slope proportional to the velocity of the condensate. Thus by choosing an initial state consisting of two ground state eigenstates of equal particle number, displaced equal and opposite amounts from the center of the trap †, we can easily isolate the phenomena that arises due solely to the collisions.

From an extensive set of simulations, in which we varied the value of  $C$ , the initial separation, and the phase difference between the two condensates, we have found that we can divide the results into two broad regimes. In the first, which we call the *linear regime*, the total wavefunction  $\Psi$  can be very well approximated at any time  $\tau$  during the first few periods of oscillation, by linearly superposing the wavefunctions that each condensate alone would have at  $\tau$ . The condensates, initially well separated, evolve to overlap at the centre of the trap producing a familiar linear fringe pattern, then separate to reconstitute themselves into two well separated condensates, regaining their original shape. The fringe spacing at any time during the overlap is determined from the  $k$  vector (i.e slope of the spatial phase) that the condensates would have at that time, if oscillating alone in the trap. The linear regime occurs when the kinetic energy of the condensates during collision dominates the nonlinear self energy, a condition that requires the condensates to be initially well separated. An approximate quantitative criteria for the linear regime can be found by considering the linear interference pattern that the two condensates would produce at the time of maximum overlap ( $\tau = \pi/2, 3\pi/2, \dots$ ). For an initial separation of the centres of the condensates of  $2d$ , then the speed of each condensate at  $\tau = \pi/2$  is  $d$ , (in the present units), giving rise to a fringe spacing of  $\pi/d$ . Approximating the separate wavefunctions by Thomas-Fermi wavefunctions (e.g [13]) the condition that the peak kinetic energy (estimated from the curvature of a  $\cos^2$  interference pattern) exceeds the peak self energy (i.e. at the centre of the fringes), and hence that the collisional behaviour is linear, takes the form

$$d > \frac{\pi}{2}(3C)^{1/3}. \quad (2)$$

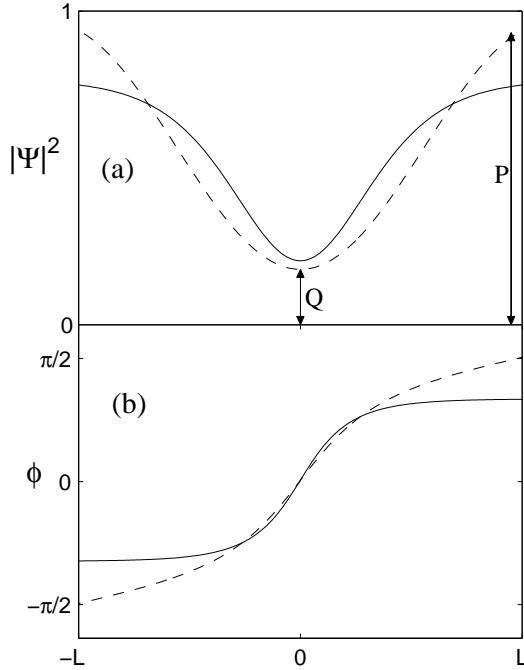
When Eq.(2) is not satisfied, the collisional behaviour falls into the *nonlinear regime*, and a representative example is given in Figure 1, where the total condensate density is shown evolving over two periods of oscillation. It is clear that the behaviour is not simple linear superposition. The initial condensates do not reform at the periodic intervals  $\tau = 2\pi, 4\pi, \dots$ , but rather, the combined condensate pulsates, contracting and expanding with period  $2\pi$ , and gradually accumulates at the centre of the trap as time progresses. The most striking feature however is the appearance of several persistent dark fringes, each with an associated phase kink of somewhat less than  $\pi$ . These dark fringes remain clearly discernible throughout the course of the simulation, surviving the extremely complex behaviour that occurs at the time of maximum contraction ( $\tau = \pi/2, 3\pi/2, \dots$ ). We identify them with gray solitons, although these have only been fully characterised in the case of an infinitely extended medium in a homogeneous environment(e.g. see [12]). In that case, they can be described analytically, and an example is shown as the solid line in Figure 2. The depth of the dip characterises the grayness of the soliton, which is said to be black (or dark) if the wave density goes

† The total wavefunction is normalised to 1, and the GP equation evolved with  $C$  equal to twice that of each eigenstate.



**Figure 1.** Evolution of condensate density for two initially stationary, displaced eigenstates for the case  $C = 2000$ . Evolution of a selected region of the phase is shown inset. Initially, phase is flat across the entire condensate, and  $d = 15$ .

to zero. The phase (Fig2(b)) has a corresponding kink in the region of the density dip, which is a sharp  $\pi$  step for a black soliton, but softens as the soliton becomes gray, thereby giving the gray soliton a velocity relative to the background density. All of these properties are possessed by the six dark fringes in Fig1 (c)-(f) , and evolve adiabatically as the background height changes. We have also shown that the fringes are very long lived (they are still present when our simulation is extended to  $\tau = 48\pi$ ) and in other circumstances we have shown that they have a repulsive interaction and survive collisions with each other. The stability of the solitons is vitally dependent on the phase kink, without which any dip quickly evolves to something quite different. The phase kink also provides a vital key to understanding the formation mechanism of these prototypical fundamental structures. The formation process begins when an essentially linear interference pattern starts forming where the two initial condensates overlap at the centre of the trap. The amplitude is small, so that nonlinear effects are negligible, and thus the spacing of these first few fringes is determined by the wavevectors (i.e. velocity) that each condensate has developed at that time. Fringes offset from the trap centre arise from the linear superposition of unequal amplitude



**Figure 2.** Comparison of (a) density and (b) phase spatial profiles for a linear interference fringe (dashed line) and the corresponding gray soliton (solid line) constrained as described in the text. Here  $P = 0.71$ ,  $Q = 0.13$ ,  $C = 2000$ ,  $L = 0.11$  and  $\zeta_f = 2.7$ .

opposing travelling waves, and thus have reduced visibility, and an associated phase kink (see Figure 2). As time progresses, the condensates continue to accelerate into each other, and as the overlap increases, additional fringes start to develop further from the center. The central few fringes evolve as local features, and remain (essentially) locked in place, retaining their initial spacing while the background density increases about them, but the new outer fringes are formed with closer spacing, appropriate to the increased velocity. As we can see from Fig 1 (b), by  $\tau = \pi/2$  (which would correspond to complete overlap in the linear case) a multitude of interference fringes have formed, and the wider spacing of the central fringes is evident. Subsequently only the central few develop into gray solitons. The key underlying physics is that a soliton is a stable structure while a (linear) interference fringe is not, but an interference fringe will only develop into a gray soliton if it is energetically favorable to do so. Otherwise, the interference fringe will disappear on a time scale less than  $\pi/2$ . To obtain a quantitative understanding we consider the energy of the two structures. A linear superposition of unequal and oppositely travelling waves of wavevector  $k$ , gives rise to a  $\cos^2$  shape which can be characterised by the height  $P$  and  $Q$  of the maximum and minimum density, respectively, as illustrated by the dotted line in Fig 2(a) (dashed line). The total energy of this wave, in the length  $2L$  between fringe peaks is

$$E_{\text{sup}} = \int_{-L}^L \Psi^*(x) H \Psi(x) dx = \frac{CL}{8} (3P^2 + 2PQ + 3Q^2) + k^2 L(P+Q). \quad (3)$$

In general a gray soliton is completely specified by two parameters, but if we now

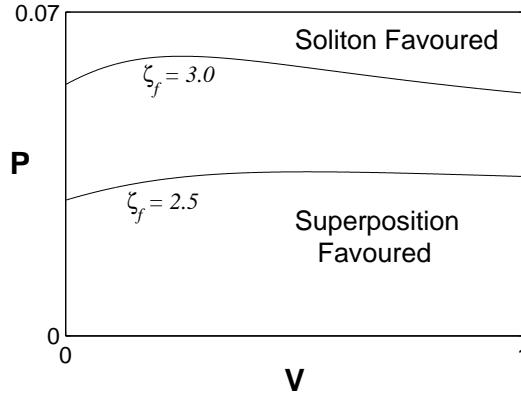
deform this same amount of condensate from the superposition, into a gray soliton within the same region  $-L < x < L$ , the constraint on  $\int_{-L}^L \Psi^*(x)\Psi(x)dx$  allows a one parameter family of solitons to be inserted in the region. This one parameter essentially determines how much of the central portion of a soliton will appear in the region, and so if we further require that the soliton has reached a specified fraction of its asymptotic value at  $x = \pm L/2$ , then only one soliton can be drawn. From the standard form of the gray soliton (e.g see Ref.([12])) the asymptotic behaviour is determined by  $\tanh(\zeta) \rightarrow 1$ , at large  $\zeta$ , (where  $\zeta$  is essentially the spatial coordinate, but renormalised in our case by a function of the parameters  $C, L, P, Q$ ). Choosing a particular value  $\zeta_f$  to make  $\tanh(\zeta_f) = f$ , some fraction close to 1, we may calculate the energy of this now completely specified soliton to be

$$E_{\text{soliton}} = \frac{CL}{4}(P+Q)^2 + \frac{4\zeta_f^2(4\zeta_f - 3)}{3CL^3} \quad (4)$$

For example choosing  $\tanh(\zeta_f) = 0.99$  gives the soliton whose density and phase we show as solid lines in Fig 2(a) and 2(b) respectively. A stable structure, the soliton, will form from the linear superposition when it is energetically favorable to do so, i.e. when  $E_{\text{soliton}} \leq E_{\text{sup}}$ . Writing  $V$  for the visibility of the superposition (i.e.  $V = (P - Q)/(P + Q)$ ) the condition of equality  $E_{\text{soliton}} = E_{\text{sup}}$ , which defines the boundary of the regime where solitons can form, can be expressed as

$$P = \frac{1+V}{6CV^2L^2} [\sqrt{9\pi^4 + 96V^2\zeta_f^2(4\zeta_f - 3)} - 3\pi^2]. \quad (5)$$

We have plotted the curve of  $P$  versus  $V$  in Figure 3 for two values of  $\zeta_f$  with illustrative values of  $C = 2000$  and  $L = \pi/7$ . A superposition formed above this



**Figure 3.** Regime where linear interference fringe develops into a gray soliton. The boundary line, Eq.(5), is plotted in terms of background density  $P$  and visibility  $V$  of the fringe, for two choices of  $\zeta_f$ .

boundary line will become a soliton, whereas a superposition formed below the line will not convert into the stable form. The parameters  $C$  and  $L$  are simply overall multiplicative factors in Eq.(5) and thus move the boundary line up or down without change in shape. It is clear that increasing the strength of the nonlinearity (i.e. increasing  $C$ ) or reducing the speed of condensates at collision (i.e increasing  $L$ ) makes

it easier for solitons to form. The choice of  $\zeta_f$  is important for quantitative purposes, and on empirical grounds, from considering a range of our simulations, we make the choice  $\zeta_f \approx 2.7$ . From Figure 3 we can then see that the central few fringes of Fig 2(b), (where  $L \approx 0.7$ ) are comfortably in the soliton regime, but as we move to the outer, narrower fringes of the condensate,  $L$  decreases, moving the boundary curve upwards, and  $P$  decreases, with the net result that solitons do not form.

The outcome of a condensate collision, particularly with regard to details of the solitons that arise, does exhibit some sensitivity to the initial phase difference of the condensates, as also noted by Reinhardt and Clark. For example, if the condensates have an initial phase difference of  $\pi$ , an odd number of solitons form, with a dark soliton at the centre of the pattern, a result easily understood in terms of the formation mechanism outlined above. There are of course other means to generate dark solitons, such as collision with a potential barrier, or even by creating a dip in a condensate eigenstate (for example by temporarily inserting a laser beam, detuned to provide a repulsive potential). In general the dip will not have the soliton shape or phase, and is thus unstable and will decompose into a number of gray solitons which will have a velocity relative to the background, according to the slope of their phase kink.

Gray solitons can also play an important role in governing the bulk motion of the condensate. They store energy (predominantly in kinetic form) in a slow moving structure, and act as a throttle to retard the movement of the condensate back to the outer region of the trap. This effect can be clearly seen in Fig. 2(e), where instead of having two well separated condensates away from the centre, as we would expect in the linear case at  $t = 4\pi$ , most of the mass has now collected at the center of the trap. We expect that solitons, (or vortices in the case of 2 or 3 spatial dimensions) will play an important role in dissipating superfluid flow.

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